

The Other Group of a Galois Extension (Castor and Pollux)*

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“Both buried now in the fertile earth though still alive. Even under the earth Zeus grants them this distinction: one day alive, the next day dead, each twin takes his turn to have status among the gods.” The Odyssey

Abstract

Let $k \subseteq K$ be a finite Galois extension of fields with Galois group G . Let \mathcal{G} be the automorphism k -group scheme of K . We construct a canonical k -subgroup scheme $\underline{G} \subset \mathcal{G}$ with the property that $\mathrm{Spec}_k(K)$ is a k -torsor for \underline{G} . \underline{G} is a constant k -group if and only if G is abelian, in which case $G = \underline{G}$.

1 Introduction

Let $k \subset K$ be a finite Galois extension of degree $n > 2$ and let $\Gamma = \mathrm{Gal}(K/k)$. Consider the automorphism group $\mathcal{G} = \mathrm{Aut}(K/k)$ as a functor from k -algebras to Γ -groups,

$$L \rightsquigarrow \mathcal{G}(L) = \mathrm{Aut}_L(L \otimes_k K).$$

As such \mathcal{G} is representable by a finite étale k -group. It turns out that \mathcal{G} can be described by a homomorphism

$$\theta : \Gamma \rightarrow \mathrm{Aut}(S_n)$$

such that $\theta(\Gamma) \subset \mathrm{Inn}(S_n)$ is a simply transitive subgroup. We can therefore write $\theta(\sigma)(g) = \beta(\sigma)g\beta(\sigma)^{-1}$ where $\beta : \Gamma \rightarrow S_n$ is a group homomorphism. It turns out that

$$G = S_n^\Gamma = \{g \in S_n \mid \beta(\sigma)g = g\beta(\sigma) \text{ for all } \sigma \in \Gamma\}$$

and

$$\beta(\Gamma) = C_{S_n}(G) = \{h \in S_n \mid hg = gh \text{ for all } g \in G\}$$

*Castor and Pollux are the two stars that give the constellation Gemini (Latin for “twins”) its name. In reality these two celestial entities are unlikely twins. Castor is a hot, white system of six stars, while Pollux is a single, large, cooler, yellow-orange star. The Chinese recognized Castor and Pollux as Yin and Yang, the two fundamental, complementary forces of the universe.

where $G = \text{Aut}_k(K) = \mathcal{G}(k) \subset \mathcal{G}(K) = S_n$. This yields an étale k -group (\underline{G}, θ) by restricting θ to obtain $\theta|_{C_{S_n}(G)} : \Gamma \rightarrow \text{Aut}(C_{S_n}(G))$, via

$$\theta(\sigma)(g) = \beta(\sigma)g\beta(\sigma)^{-1}.$$

for $g \in C_{S_n}(G)$ and $\sigma \in \Gamma$. (\underline{G}, θ) is thus a k -subgroup of (S_n, θ) .

Theorem 1.1. *Let $X = \text{Spec}_k(K)$. The restriction*

$$\underline{G} \times_k X \rightarrow X$$

makes X into a principal homogeneous space for \underline{G} .

Theorem 1.2. *\underline{G} is a constant k -group if and only if G is abelian.*

2 Simply Transitive Groups

In this section we assemble a few basic results about finite groups that are needed for our main results.

Consider the finite group G of order n thought of as a group of transformations acting on itself by either left translations or right translations. We write $l_g(x) = gx$ and $r_g(x) = xg$. If we think of G acting on itself by left translations we write $G^l = \{l_g\} \subseteq S_n$. If we think of G acting on itself by right translations we write $G^r = \{r_g\} \subseteq S_n$. We write G if we think of G as the set being acted on somehow, or if we think of G acting somehow other than on itself by right or left translation. Notice that $IG^lI^{-1} = G^r$ where $I \in S_n$ is defined by $I(g) = g^{-1}$. Indeed, for any $g \in G$, $I(l_g)I^{-1} = r_{g^{-1}}$.

Proposition 2.1. *$C_{S_n}(G^l) = G^r$ and $C_{S_n}(G^r) = G^l$. Furthermore $G^l \cap G^r \cong Z(G)$, the center of G .*

Proof. It is straightforward to check that $G^r \subseteq C_{S_n}(G^l)$. Suppose conversely that $f \in C_{S_n}(G^l)$ and let $g = f(1) \in G$. Then, for $x \in G$, $f(x) = f(x1) = (f \circ l_x)(1) = (l_x \circ f)(1) = xf(1) = xg$. Evidently $f = r_g \in G^r$.

Suppose that $f \in G^l \cap G^r$. Then $f = l_g = r_h$ for $g, h \in G$. But then $f(1) = g = l_g(1) = r_h(1) = h$. On the other hand, for any $x \in G$, $gx = l_g(x) = f(x) = r_h(x) = xh$. Conversely, it is trivial to check that $Z(G^r) \cup Z(G^l) \subset G^r \cap G^l$. \square

The following Proposition will be useful in the next section.

Proposition 2.2. *Suppose that $H < G^r$ is a subgroup of order k so that $n = km$. Then $|C_{S_n}(H)| = (m!)k^m$.*

Proof. $H = \{r_g \mid g \in H\}$. So let

$$G_r = \bigsqcup_{i=1}^m x_i H = \bigsqcup_{i=1}^m A_i.$$

Let

$$\sigma \in S_n \text{ and } y_i \in A_i, i = 1, \dots, m. \quad (*)$$

Define a bijection $f_* : G \rightarrow G$ by setting $f_*(x_i h) = y_{\sigma(i)} h$ so that $f_*(A_i) = A_{\sigma(i)}$. Then, for $g \in H$,

$$f_* \circ r_g(x_i h) = f_*(x_i h g) = y_{\sigma(i)} h g = r_g(y_{\sigma(i)} h) = r_g \circ f_*(x_i h).$$

Conversely, given $f \in C_{S_n}(H)$, let $f(x_i) = y_{\sigma(i)} \in A_{\sigma(i)}$. One checks that if $i \neq j$ then $\sigma(i) \neq \sigma(j)$, since f is injective. But then $f(x_i h) = y_{\sigma(i)} h$ where $\sigma(i) \in S_m$. Thus f is among those already listed in $(*)$. Counting these all up we obtain that there are $(m!)k^m$ different possibilities. \square

Corollary 2.3. *Suppose that $m > 1$ and $n > 2$. Let $H < G^r$ be such that $C_{S_n}(H) = G^l$. Then $H = G^r$.*

Proof. If $m > 1$ and $n > 2$ then $(m!)k^m > n$. \square

Definition 2.4. A subgroup $G \subset S_n$ is called *simply transitive* if G acts simply and transitively on the set $\{1, 2, \dots, n\}$ for some transitive action of S_n on $\{1, 2, \dots, n\}$.

Remark 2.5. One might wonder in what sense this transitive action is well-defined. If $n = 2$ then $G = S_n$. So that case is alright. If $n \neq 6$ then the action of S_n on $\{1, 2, \dots, n\}$ is well-defined up to inner automorphism, so this case is also fine; one just takes the action of G to be conjugation on the n conjugate subgroups of S_n of index n . However there is a nontrivial outer automorphism of S_6 . See [1]. Correspondingly, there are two conjugacy classes of subgroups of S_6 isomorphic to S_5 and, consequently, there are two different ways to define the cycle structure on the elements of S_6 . Let $G \subset S_6$ be a subgroup that is simply transitive on $\{1, 2, 3, 4, 5, 6\}$ for the transitive action T_1 of S_6 on $\{1, 2, 3, 4, 5, 6\}$. Now G is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ or S_3 . Let $g \in G$ be an element of order two. Since G is simply transitive for the T_1 action, g has cycle structure $(2, 2, 2)$ for this action. But it is known (see [1]) that, for the “other” transitive action T_2 of S_6 on $\{1, 2, 3, 4, 5, 6\}$, g must have cycle structure $(1, 1, 1, 1, 2)$. So g has four fixed points for the T_2 action. We conclude that, if $G \subset S_6$ is simply transitive for one structure, then it is not simply transitive for the other (nonconjugate) structure.

3 Finite Étale k -groups

Let k be a field and let Γ be the Galois group of the separable closure k_s over k . We first remind the reader of the well-known equivalence between finite Γ -sets and finite separable k -algebras.

If $\Gamma \times X \rightarrow X$ is a Γ -set we denote the action of $\sigma \in \Gamma$ on $x \in X$ by $\sigma(x)$. If $\Gamma \times X \rightarrow X$ is a finite Γ -set we define

$$A_X = \text{Hom}_s(X, k_s)^\Gamma$$

where the action of Γ on $\text{Hom}_s(X, k_s)$ is given by $\sigma(f)(x) = \sigma(f(\sigma^{-1}(x)))$. Here σ also acts on k_s and “ $\text{Hom}_s(U, V)$ ” denotes the set of functions from U to V . One checks that A_X is a finite separable k -algebra.

If $k \rightarrow A$ is a finite separable k -algebra we define

$$X_A = \text{Hom}_k(A, k_s)$$

where the action of Γ on X_A is given by its action on k_s . i.e. $\sigma(x)(a) = \sigma(x(a))$. “ $\text{Hom}_k(A, B)$ ” denotes the set of k -algebra homomorphisms from A to B .

There is a canonical isomorphism of Γ -algebras,

$$A \otimes_k k_s \cong \text{Hom}_s(X_A, k_s),$$

defined by $a \otimes \lambda \rightsquigarrow f$, where $f(x) = x(a)\lambda$.

Theorem 3.1. *The functors $A \rightsquigarrow A_X$ and $X \rightsquigarrow X_A$ determine an equivalence between the category of finite separable k -algebras and the category of finite Γ -sets.*

Proof. See Theorem 6.4 of [2]. □

With this in mind we make the following definition.

Definition 3.2. A *finite étale k -group* (\underline{G}, β) is a finite abstract group G together with a homomorphism of groups $\beta : \Gamma \rightarrow \text{Aut}(G)$. The k -Hopf algebra of functions on \underline{G} is $k[\underline{G}] = \{f : G \rightarrow k_s \mid f(\beta(\sigma)(g)) = \beta(\sigma)(f(g)) \text{ for all } \sigma \in \Gamma\}$. G is called a Γ -group.

Using Theorem 3.1 we obtain the following result.

Theorem 3.3. *There is a canonical equivalence between the category of finite Γ -groups and the category of finite separable k -Hopf algebras.*

4 The Other Group

Let $k \subset K$ be a finite Galois extension of degree n with Galois group G , and let $\Gamma = \text{Gal}(k_s/k)$ be the Galois group of the separable closure k_s of k . Define a finite étale k -group by the rule

$$\mathcal{G}(L) = \text{Aut}_L(L \otimes_k K)$$

where $k \subset L$ is a k -algebra. Notice that $\mathcal{G}(k_s) \cong S_n$. By Theorem 3.3 there is a homomorphism of groups

$$\theta : \Gamma \rightarrow \text{Aut}(S_n)$$

such that $S_n^\Gamma = G \subset S_n$. Furthermore, $G \subset S_n$ is a simply transitive subgroup in the sense of Definition 2.4.

Proposition 4.1. *Assume $n > 2$. Then the group homomorphism*

$$\theta : \Gamma \rightarrow \text{Aut}(S_n)$$

factors through $\text{int} : S_n \rightarrow \text{Aut}(S_n)$. Thus Γ acts on S_n by

$$g \rightsquigarrow \beta(\sigma)g\beta(\sigma)^{-1}$$

where $\beta : \Gamma \rightarrow S_n$ is a group homomorphism.

Proof. There is no doubt here unless $n = 6$. Now $G \subset S_6$, being simply transitive, must be isomorphic to $\mathbb{Z}/6\mathbb{Z}$ or S_3 , and in such a way that any element $g \in G = S_6^\Gamma$ of order two is conjugate in S_6 to $(12)(34)(56)$ (see Remark 2.4). But it is known [1] that any outer automorphism of S_6 exchanges the conjugacy class of $(12)(34)(56)$ with that of (12) . Thus, if $\sigma \in \Gamma$ and $\theta(\sigma)$ is outer then $\theta(\sigma)(g) \neq g$. Thus $\theta(\sigma)$ is inner for all $\sigma \in \Gamma$. \square

Corollary 4.2. *Assume $n > 2$. Suppose that $\theta : \Gamma \rightarrow \text{Aut}(S_n)$ is such that $G = S_n^\Gamma$ is a simply transitive subgroup. Then*

1. $\theta(\Gamma) \subseteq \text{Inn}(S_n)$.
2. $\theta(\Gamma) = C_{S_n}(G)$.

Proof. Corollary 2.3 and Proposition 4.1. \square

Let $k \subset K$ be a finite Galois extension of degree $n > 2$ and let $\Gamma = \text{Gal}(K/k)$. Consider the Galois group $\mathcal{G} = \text{Gal}(K/k)$ as a functor from k -algebras to Γ -groups,

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As such \mathcal{G} is representable by a finite étale k -group. As above \mathcal{G} can be described by a homomorphism

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such that $\theta(\Gamma) \subset \text{Inn}(S_n)$. We can therefore write $\theta(\sigma)(g) = \beta(\sigma)g\beta(\sigma)^{-1}$ where $\beta : \Gamma \rightarrow S_n$ is a group homomorphism. Furthermore $\beta(\Gamma) \subset S_n$ is a simply transitive subgroup. Now,

$$G = S_n^\Gamma = \{g \in S_n \mid \beta(\sigma)g = g\beta(\sigma) \text{ for all } \sigma \in \Gamma\}$$

and

$$\theta(\Gamma) = C_{S_n}(G) = \{h \in S_n \mid hg = gh \text{ for all } g \in G\}.$$

where $G = \text{Aut}_k(K) = \mathcal{G}(k) \subset \mathcal{G}(K) = S_n$. This yields an étale k -group (\underline{G}, θ) by restricting θ to obtain $\theta|_{C_{S_n}(G)} : \Gamma \rightarrow \text{Aut}(C_{S_n}(G))$, via

$$\theta(\sigma)(g) = \beta(\sigma)g\beta(\sigma)^{-1}.$$

for $g \in C_{S_n}(G)$. (\underline{G}, θ) is thus a k -subgroup of (S_n, θ) . It is the centralizer in (S_n, θ) of (G, θ) , the latter being a constant subgroup since $G = S_n^\Gamma$.

Theorem 4.3. *Let $X = \text{Spec}_k(K)$. The restriction*

$$\underline{G} \times_k X \rightarrow X$$

makes X into a principal homogeneous space for \underline{G} .

Proof. $\underline{G}(K) \times X(K) \rightarrow X(K)$ acts simply and transitively by Proposition 2.1. Thus the map $\underline{G}(K) \times X(K) \rightarrow X(K) \times X(K)$, $(g, x) \rightarrow (gx, x)$, is a bijection. \square

Theorem 4.4. *$\underline{G} \cap G = Z(G)$, the center of G . \underline{G} is a constant k -group if and only if G is abelian. In this case $\underline{G} = G$.*

Proof. See Proposition 2.1. \square

References

- [1] P. J. Lorimer, *The Outer Automorphisms of S_6* , American Mathematical Monthly, Vol.73, No.6, 1966, 642-643.
- [2] W. C. Waterhouse, *“Introduction to affine group schemes”*, Graduate Texts in Math. 66, Springer-Verlag, New York, 1979.

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